International Journal of Theoretical Physics, Vol. 21, Nos. 8/9, 1982

Pretty Mathematics[1]

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Received July 10, 1981

Mathematical beauty is important in particle theories, two of which are described.

A good deal of my research work in physics has consisted in not setting out to solve some particular problem, but simply examining mathematical quantities of a kind that physicists use and trying to fit them together in an interesting way regardless of any application that the work may have. It is simply a search for pretty mathematics. It may turn out later that the work does have an application. Then one has had good luck.

I can give a good example of this procedure. At one time, in 1927, I was playing around with three 2×2 matrices whose squares are equal to unity and which anticommute with one another. Calling them $\sigma_1, \sigma_2, \sigma_3$, I noticed that if one multiplied them into the three components of a momentum so as to form $\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3$, one obtained a quantity whose square was just $p_1^2 + P_2^2 + p_3^2$. This was an exciting result, but what use could one make of it?

One could use $\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3$ as the Hamiltonian in a Schrödinger wave equation, giving the wave function two components so that the σ matrices can be applied to it. One then had a relativistic wave equation. But it applied only to a particle of zero rest mass. To get a theory for a particle with nonzero rest mass one would need four σ matrices anticommuting with one another, and such matrices did not exist. So my work was of no use for the electron, which was what I was mainly interested in. I therefore had to abandon it.

It was not until some weeks later that I realized there is no need to restrict oneself to 2×2 matrices. One could go on to 4×4 matrices, and the

[1]Presented at the Dirac Symposium, Loyola University, New Orleans, May 1981.

0020-7748/82/0800-0603$03.00/0

problem is then easily soluble. In retrospect, it seems strange that one can be so much held up over such an elementary point.

The resulting wave equation for the electron turned out to be very successful. It led to correct values for the spin and the magnetic moment. This was quite unexpected. The work all followed from a study of pretty mathematics, without any thought being given to these physical properties of the electron.

Another example of pretty mathematics led to the idea of the magnetic monopole. When I did this work I was hoping to find some explanation of the fine-structure constant $\hbar c/e^2$. But this failed. The mathematics led inexorably to the monopole.

From the theoretical point of view one would think that monopoles should exist, because of the prettiness of the mathematics. Many attempts to find them have been made, but all have been unsuccessful. One should conclude that pretty mathematics by itself is not an adequate reason for nature to have made use of a theory. We still have much to learn in seeking for the basic principles of nature.

I would like to discuss a further development of pretty mathematics that is provided by a relativistic theory of a particle that I put forward in 1970. Here the wave function has only one component, instead of the usual four, but the particle has internal degrees of freedom, consisting of two harmonic oscillators. Call the coordinates of these oscillators q_1, q_2, and the conjugate momenta q_3, q_4, so the four q's satisfy the commutation relations (with $\hbar=1$)

$$q_a q_b - q_b q_a = \beta_{ab}, \qquad a, b = 1,2,3,4$$

where β is the matrix

$$\beta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Note that β is skew and $\beta^2 = -1$.

We then consider the four q's as a column matrix, so that q appears as a column matrix with four elements. We set up the wave equation

$$\left\{ \frac{\partial}{\partial t} + \alpha_r \frac{\partial}{\partial x^r} + \beta \right\} q\psi = 0 \tag{1}$$

where the α's are 4×4 matrices whose squares are unity and which anticommute with one another and with β. Also the α's must be chosen to have only real elements. This wave equation is formally very similar to the usual electron wave equation, with the difference that we have the column matrix $q\psi$, with just one ψ, replacing the four components of the usual ψ.

There are four equations in (1), corresponding to the four components of $q\psi$. Of these, it turns out that only three are independent. For one function ψ to satisfy three independent equations it is necessary that certain consistency conditions shall be fulfilled. One finds that they are fulfilled, provided ψ satisfies the de Broglie equation (with $m = 1$) for all values of the internal coordinates. One finds also that the equation is relativistic, and that the mass has to be positive. These are pretty results, and lead one to wonder if the theory can describe any particle in nature.

One cannot assign a charge to the particle and let it interact with the electromagnetic field in the usual way, by making

$$i\hbar\frac{\partial}{\partial x^\mu} \to i\hbar\frac{\partial}{\partial x^\mu} + eA\mu \tag{2}$$

because then the equations cease to be consistent. The consistency conditions are so restrictive that there is very little one can do with the theory, and I did not find any way of developing it so as to lead to a hope of applying it.

Recently, an extension of the theory has been made by Sudarshan and his co-workers. They extend the internal degrees of freedom of the particle by replacing

$$q_a \to \sum_r \xi_a^r$$

for each of the four values of a. For each r the ξ_a^r are like the original q_a describing two harmonic oscillators, but the ξ_a^r for one value of r anticommute with those for another value of r. Under these conditions Sudarshan shows that the equations are still consistent and still provide a reasonable relativistic theory.

This is truly a remarkable result. It makes the theory much more flexible. It becomes possible to introduce interaction with the electromagnetic field in the usual way, according to the procedure (2). It renews hope that the theory will have an application to some particle in nature, but a good deal more work will first have to be done on it.

With an approach like this to a new theory, there is a chance of new features appearing which one could never find by just making straightforward developments of the old theories.

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